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HIGHER HAMMING WEIGHTS FOR LOCALLY RECOVERABLE CODES ON ALGEBRAIC CURVES

Edoardo Ballico ^{*}& Chiara Marcolla¹

ABSTRACT

We study locally recoverable codes on algebraic curves. In the first part of the manuscript, we provide a bound on the generalized Hamming weight of these codes. In the second part, we propose a new family of algebraic geometric LRC codes, which are LRC codes from the Norm-Trace curve. Finally, using some properties of Hermitian codes, we improve the bounds on the distance proposed in [1] of some Hermitian LRC codes.

1 INTRODUCTION

The v -th generalized Hamming weight $d_v(C)$ of a linear code C is the minimum support size of v -dimensional subcodes of C . The sequence $d_1(C), \dots, d_k(C)$ of generalized Hamming weights was introduced by Wei [37] to characterize the performance of a linear code on the wire-tap channel of type II. Later, the GHWs of linear codes have been used in many other applications regarding the communications, as for bounding the covering radius of linear codes [15], in network coding [26], in the context of list decoding [7, 9], and finally for secure secret sharing [18]. Moreover, in [2] the authors show in which way an arbitrary linear code gives rise to a secret sharing scheme, in [16, 17] the connection between the trellis or state complexity of a code and its GHWs is found and in [4] the author proves the equivalence to the dimension/length profile of a code and its generalized Hamming weight. For these reasons, the GHWs (and their *extended* version, the *relative* generalized Hamming weights [21, 19]) play a central role in coding theory. In particular, generalized and relative generalized Hamming weights are studied for Reed-Muller codes [10, 23] and for codes constructed by using an algebraic curve [6]

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as Goppa codes [24, 38], Hermitian codes [12, 25] and Castle codes [27].

In this paper, we provide a bound on the generalized Hamming weight of locally recoverable codes on the algebraic curves proposed in [1]. Moreover, we introduce a new family of algebraic geometric LRC codes and improve the bounds on the distance for some Hermitian LRC codes.

Locally recoverable codes were introduced in [8] and they have been significantly studied because of their applications in distributed and cloud storage systems [3, 13, 32, 34, 35]. We recall that a code $C \in (\mathbb{F}_q)^n$ has locality r if every symbol of a codeword c can be recovered from a subset of r other symbols of c .

In other words, we consider a finite field $K = \mathbb{F}_q$, where q is a power of a prime, and an $[n, k]$ code C over the field K , where $k = \log_q(|C|)$. For each $i \in \{1, \dots, n\}$ and each $a \in K$ set $C(i, a) = \{c \in C \mid c_i = a\}$. For each $I \subseteq \{1, \dots, n\}$ and each $S \subseteq C$ let S_I be the restriction of S to the coordinates in I .

Definition 1. Let C be an $[n, k]$ code over the field K , where $k = \log_q(|C|)$. Then C is said to have **all-symbol locality r** if for each $a \in \mathbb{F}_q$ and each $i \in \{1, \dots, n\}$ there is $I_i \subset \{1, \dots, n\} \setminus \{i\}$ with $|I_i| \leq r$, such that for $C_{I_i}(i, a) \cap C_{I_i}(i, a') = \emptyset$ for all $a \neq a'$. We use the notation (n, k, r) to refer to the parameters of this code.

Note that if we receive a codeword c correct except for an erasure at i , we can recover the codeword by looking at its coordinates in I_i . For this reason, I_i is called a *recovering set* for the symbol c_i .

Let C be an (n, k, r) code, then the distance of this code has to verify the bound proved in [28, 8] that is $d \leq n - k - \lceil k/r \rceil + 2$. The codes that achieve this bound with equality are called *optimal* LRC codes [32, 34, 35]. Note that when $r = k$, we obtain the Singleton bound, therefore optimal LRC codes with $r = k$ are MDS codes.

LAYOUT OF THE PAPER This paper is divided as follows. In Section 2 we recall the notions of algebraic geometric codes and the definition of algebraic geometric locally recoverable codes introduced in [1]. In Section 3 we provide a bound on the generalized Hamming weights of the latter codes. In Section 4 we propose a new family of algebraic geometric LRC codes, which are LRC codes from the Norm–Trace curve. Finally, in Section 5 we improve the bounds on the distance proposed in [1] for some Hermitian LRC codes, using some properties of the Hermitian codes.

2 PRELIMINARY NOTIONS

2.1 Algebraic geometric codes

Let $K = \mathbb{F}_q$ be a finite field, where q is a power of a prime. Let \mathcal{X} be a smooth projective absolutely irreducible nonsingular curve over K . We denote by $K(\mathcal{X})$ the rational func-

tions field on \mathcal{X} . Let D be a divisor on the curve \mathcal{X} . We recall that the *Riemann-Roch space* associated to D is a vector space $\mathcal{L}(D)$ over K defined as

$$\mathcal{L}(D) = \{f \in K(\mathcal{X}) \mid (f) + D \geq 0\} \cup \{0\}.$$

where we denote by (f) the divisor of f .

Assume that P_1, \dots, P_n are rational points on \mathcal{X} and D is a divisor such that $D = P_1 + \dots + P_n$. Let G be some other divisor such that $\text{supp}(D) \cap \text{supp}(G) = \emptyset$. Then we can define the algebraic geometric code as follows:

Definition 2. The **algebraic geometric code** (or AG code) $C(D, G)$ associated with the divisors D and G is defined as

$$C(D, G) = \{(f(P_1), \dots, f(P_n)) \mid f \in \mathcal{L}(G)\} \subset K^n.$$

The dual $C^\perp(D, G)$ of $C(D, G)$ is an algebraic geometric code.

In other words an algebraic geometric code is the image of the evaluation map $\text{Im}(ev_D) = C(D, G)$, where the *evaluation map* $ev_D : \mathcal{L}(G) \rightarrow K^n$ is given by

$$ev_D(f) = (f(P_1), \dots, f(P_n)) \in K^n.$$

Note that if $D = P_1 + \dots + P_n$ and we denote by $\mathcal{P} = \{P_1, \dots, P_n\}$ we can also indicate ev_D as $ev_{\mathcal{P}}$.

2.2 Algebraic geometric locally recoverable codes

In this section we consider the construction of algebraic geometric locally recoverable codes of [1].

Let \mathcal{X} and \mathcal{Y} be smooth projective absolutely irreducible curves over K . Let $g : \mathcal{X} \rightarrow \mathcal{Y}$ be a rational separable map of curves of degree $r + 1$. Since g is separable, then there exists a function $x \in K(\mathcal{X})$ such that $K(\mathcal{X}) = K(\mathcal{Y})(x)$ and that x satisfies the equation $x^{r+1} + b_r x^r + \dots + b_0 = 0$, where $b_i \in K(\mathcal{Y})$. The function x can be considered as a map $x : \mathcal{X} \rightarrow \mathbb{P}_K$. Let $h = \deg(x)$ be the degree of x .

We consider a subset $S = \{P_1, \dots, P_s\} \subset \mathcal{Y}(K)$ of \mathbb{F}_q -rational points of \mathcal{Y} , a divisor Q_∞ such that $\text{supp}(Q_\infty) \cap \text{supp}(S) = \emptyset$ and a positive divisor $D = tQ_\infty$. We denote by

$$\mathcal{A} = g^{-1}(S) = \{P_{ij}, \text{ where } i = 0, \dots, r, j = 1, \dots, s\} \subset \mathcal{X}(K),$$

where $g(P_{ij}) = P_i$ for all i, j and assume that b_i are functions in $\mathcal{L}(n_i Q_\infty)$ for some natural numbers n_i with $i = 1, \dots, r$.

Let $\{f_1, \dots, f_m\}$ be a basis of the Riemann-Roch space $\mathcal{L}(D)$. By the Riemann-Roch Theorem we have that $m \geq \deg(D) + 1 - g_{\mathcal{Y}}$, where $g_{\mathcal{Y}}$ is the genus of \mathcal{Y} .

From now on, we assume that $m = \deg(D) + 1 - g_{\mathcal{Y}}$, where $\deg(D) = t\ell$, and we consider the K -subspace V of $K(\mathcal{X})$ of dimension rm generated by

$$\mathcal{B} = \{f_j x^i, i = 0, \dots, r-1, j = 1, \dots, m\}.$$

We consider the evaluation map $\text{ev}_{\mathcal{A}} : V \rightarrow K^{(r+1)s}$. Then we have the following theorem.

Theorem 1. *The linear space $C(D, g) = \text{Span}_{K^{(r+1)s}} \langle \text{ev}_{\mathcal{A}}(\mathcal{B}) \rangle$ is an (n, k, r) algebraic geometric LRC code with parameters*

$$\begin{aligned} n &= (r+1)s \\ k &= rm \geq r(\text{tl} + 1 - g_y) \\ d &\geq n - \text{tl}(r+1) - (r-1)h. \end{aligned}$$

Proof. See Theorem 3.1 of [1]. □

The AG LRC codes have an additional property. They are LRC codes (n, k, r) with $(r+1) \mid n$ and $r \mid k$. The set $\{1, \dots, n\}$ can be divided into $n/(r+1)$ disjoint subsets U_j for $1 \leq j \leq s$ with the same cardinality $r+1$. For each i the set $I_i \subseteq \{1, \dots, n\} \setminus \{i\}$ is the complement of i in the element of the partition U_j containing j , i.e. for all $i, j \in \{1, \dots, n\}$ either $I_i = I_j$ or $I_i \cap I_j = \emptyset$.

Moreover, they have also the following nice property. Fix $w \in (K)^n$ and denote by $w_{U_j} = \{w_{\iota}, \text{ for any } \iota \in U_j\}$. Suppose we receive all the symbols in U_j . There is a simple linear parity test on the $r+1$ symbols of U_j such that if this parity check fails we know that at least one of the symbols in U_j is wrong. If we are guaranteed (or we assume) that at most one of the symbols in U_j is wrong and the parity check is OK, then all the symbols in U_j are correct. Moreover we can recover an erased symbol w_{ι} , with $\iota \in U_j$ using a polynomial interpolation through the points of the recovering set w_{U_j} .

3 GENERALIZED HAMMING WEIGHTS OF AG LRC CODES

Let K be a field and let \mathcal{X} be a smooth and geometrically connected curve of genus $g \geq 2$ defined over the field K . We also assume $\mathcal{X}(K) \neq \emptyset$. We recall the following definitions:

Definition 3 ([29], [30]). The **K-gonality** $\gamma_K(\mathcal{X})$ of \mathcal{X} over a field K is the smallest possible degree of a dominant rational map $\mathcal{X} \rightarrow \mathbb{P}_K^1$. For any field extension L of K , we define also the **L-gonality** $\gamma_L(\mathcal{X})$ of \mathcal{X} as the gonality of the base extension $\mathcal{X}_L = \mathcal{X} \times_K L$. It is an invariant of the function field $L(\mathcal{X})$ of \mathcal{X}_L .

Moreover, for each integer $i > 0$, the *i-th gonality* $\gamma_{i,L}(\mathcal{X})$ of \mathcal{X} is the minimal degree z such that there is $R \in \text{Pic}^z(\mathcal{X})(L)$ with $h^0(R) \geq i+1$. The sequence $\gamma_{i,\bar{K}}(\mathcal{X})$ is the usual *gonality sequence* [20]. Moreover, the integer $\gamma_{1,K}(\mathcal{X}) = \gamma_K(\mathcal{X})$ is the K-gonality of \mathcal{X} .

Let $K = \mathbb{F}_q$ a finite field with q elements. Let $C \subset K^n$ be a linear $[n, k]$ code over K . We recall that the *support* of C is defined as follows

$$\text{supp}(C) = \{i \mid c_i \neq 0 \text{ for some } c \in C\}.$$

So $\#\text{supp}(C)$ is the number of nonzero columns in a generator matrix for C . Moreover, for any $1 \leq v \leq k$, the *v-th generalized Hamming weight* of C [14, §7.10], [36, §1.1] is defined by

$$d_v(C) = \min\{\#\text{supp}(\mathcal{D}) \mid \mathcal{D} \text{ is a linear subcode of } C \text{ with } \dim(\mathcal{D}) = v\}.$$

In other words, for any integer $1 \leq v \leq k$, $d_v(C)$ is the v -th minimum support weights, i.e. the minimal integer t such that there are an $[n, v]$ subcode \mathcal{D} of C and a subset $S \subset \{1, \dots, n\}$ such that $\#(S) = t$ and each codeword of \mathcal{D} has zero coordinates outside S . The sequence $d_1(C), \dots, d_k(C)$ of generalized Hamming weights (also called *weight hierarchy* of C) is strictly increasing (see Theorem 7.10.1 of [14]). Note that $d_1(C)$ is the minimum distance of the code C .

Let us consider \mathcal{X} and \mathcal{Y} smooth projective absolutely irreducible curves over K and let $g : \mathcal{X} \rightarrow \mathcal{Y}$ be a rational separable map of curves of degree $r + 1$. Moreover we take $r, t, Q_\infty, f_1, \dots, f_m$ and $\mathcal{A} = g^{-1}(S)$ defined as Section 2.2. So we can construct an (n, k, r) algebraic geometric LRC code C as in Theorem 1. For this code we have the following:

Theorem 2. *Let C be an (n, k, r) algebraic geometric LRC code as in Theorem 1. For every integer $v \geq 2$ we have that*

$$d_v(C) \geq n - t\ell(r + 1) - (r - 1)h + \gamma_{v-1, K}(\mathcal{X}).$$

Proof. Take a v -dimensional linear subspace \mathcal{D} of C and call

$$E \subseteq \{P_{ij} \mid i = 0, \dots, r, j = 1, \dots, s\},$$

the set of common zeros of all elements of \mathcal{D} . Since $n - d_v(C) = \#(E)$, we have to prove that $t\ell(r + 1) + (r - 1)h - \#(E) \geq \gamma_{v-1, K}(\mathcal{X})$. Fix $u \in \mathcal{D} \setminus \{0\}$ and let F_u denote the zeros of u . Note that F_u is contained in the set $\{P_{ij} \mid i = 0, \dots, r, j = 1, \dots, s\}$ by the definition of the code C . We have $F_u \supseteq E$. By the definition of the integers t, ℓ and $h := \deg(x)$, we have $\#(F_u) \leq t\ell(r + 1) + (r - 1)h$. The divisors $F_u - E$, $u \in \mathcal{D} \setminus \{0\}$ form a family of linearly equivalent non-negative divisors, each of them defined over K . Since $\dim(\mathcal{D}) = v$, the definition of $\gamma_{v-1, \bar{K}}(\mathcal{X})$ gives $\#(F_u) - \#(E) \geq \gamma_{v-1, K}(\mathcal{X})$. This inequality for a single $u \in \mathcal{D} \setminus \{0\}$ proves the theorem. \square

See Remark 1 for an application of Theorem 2.

4 LRC CODES FROM NORM-TRACE CURVE

In this section we propose a new family of Algebraic Geometric LRC codes, that is, a LRC codes from the Norm-Trace curve. Moreover, we compute the \mathbb{F}_{q^u} -gonality of the Norm-Trace curve.

Let $K = \mathbb{F}_{q^u}$ be a finite field, where q is a power of a prime. We consider the *norm* $N_{\mathbb{F}_q}^{\mathbb{F}_{q^u}}$ and the *trace* $\text{Tr}_{\mathbb{F}_q}^{\mathbb{F}_{q^u}}$, two functions from \mathbb{F}_{q^u} to \mathbb{F}_q defined as

$$N_{\mathbb{F}_q}^{\mathbb{F}_{q^u}}(x) = x^{1+q+\dots+q^{u-1}} \text{ and } \text{Tr}_{\mathbb{F}_q}^{\mathbb{F}_{q^u}}(x) = x + x^q + \dots + x^{q^{u-1}}.$$

The *Norm-Trace curve* χ is the curve defined over K by the following affine equation

$$N_{\mathbb{F}_q}^{\mathbb{F}_{q^u}}(x) = \text{Tr}_{\mathbb{F}_q}^{\mathbb{F}_{q^u}}(y),$$

that is,

$$x^{(q^u-1)/(q-1)} = y^{q^{u-1}} + y^{q^{u-2}} + \dots + y \text{ where } x, y \in K \quad (1)$$

The Norm-Trace curve χ has exactly $n = q^{2u-1}$ K -rational affine points (see Appendix A of [5]), that we denote by $\mathcal{P}_\chi = \{P_1, \dots, P_n\}$. The genus of χ is $g = \frac{1}{2}(q^{u-1} - 1)(\frac{q^u-1}{q-1} - 1)$. Note that if we consider $u = 2$, we obtain the Hermitian curve.

Starting from the Norm-Trace curve, we have two different ways to construct Norm-Trace LRC codes.

PROJECTION ON x We have to construct a q^u -ary (n, k, r) LRC codes. We consider the natural projection $g(x, y) = x$. Then the degree of g is $q^{u-1} = r + 1$ and the degree of y is $h = 1 + q + \dots + q^{u-1}$.

To construct the codes we consider $S = \mathbb{F}_{q^u}$ and $D = tQ_\infty$ for some $t \geq 1$. Then, using a construction of Theorem 1 we find the parameters for these Norm-Trace LRC codes.

Proposition 1. *A family of Norm-Trace LRC codes has the following parameters:*

$$n = q^{2u-1}, \quad k = mr = (t + 1)(q^{u-1} - 1)$$

and

$$d \geq n - tq^{u-1} - (q^{u-1} - 1)(1 + q + \dots + q^{u-1}).$$

PROJECTION ON y We have to construct a q^u -ary (n, k, r) LRC codes. We consider the other natural projection $g'(x, y) = y$. Then $\deg(g') = 1 + q + \dots + q^{u-1} = r + 1$. In this case we take $S = \mathbb{F}_{q^u} \setminus M$, where

$$M = \{a \in \mathbb{F}_{q^u} \mid a^{q^{u-1}} + a^{q^{u-2}} + \dots + a = 0\},$$

so $r = q + \dots + q^{u-1}$ and $h = \deg(x) = q^{u-1}$. Then, using Theorem 1 we have the following

Proposition 2. *A family of Norm-Trace LRC codes has the following parameters:*

$$n = q^{2u-1} - q^{u-1}, \quad k = mr = (t + 1)(q + \dots + q^{u-1})$$

and

$$d \geq n - tq^{u-1} - (q + \dots + q^{u-1}) - q^{u-1}(q^{u-1} + \dots + q - 1).$$

For the Norm-Trace curve χ we are able to find the K -gonality of χ .

Lemma 1. *Let χ be a Norm-Trace curve defined over \mathbb{F}_{q^u} , where $u \geq 2$. We have $\gamma_{1, \mathbb{F}_{q^u}}(\chi) = q^{u-1}$.*

Proof. The linear projection onto the x axis has degree q^{u-1} and it is defined over \mathbb{F}_q and hence over \mathbb{F}_{q^u} . Thus $\gamma_{1, \mathbb{F}_{q^u}}(\chi) \leq q^{u-1}$. Denote by $z = \gamma_{1, \mathbb{F}_{q^u}}(\chi)$ and assume that $z \leq q^{u-1} - 1$. By the definition of K -gonality, there is a non-constant morphism $w: \chi \rightarrow \mathbb{P}^1$ with $\deg(w) = z$ and defined over \mathbb{F}_{q^u} . Since $w(\chi(\mathbb{F}_{q^u})) \subseteq \mathbb{P}^1(\mathbb{F}_{q^u})$, we get $\sharp(\chi(\mathbb{F}_{q^u})) \leq z(q^u + 1) \leq (q^{u-1} - 1)(q^u + 1)$, that is a contradiction. \square

Remark 1. By Lemma 1, we can apply Theorem 2 to the Norm-Trace curve. In fact, we can consider the gonality sequence over K of χ to get a lower bound on the second generalized Hamming weight of the two families of Norm-Trace LRC codes:

- Let $t \geq 1$ and let C be a $(q^{2u-1}, (t+1)(q^{u-1}-1), q^{u-1}-1)$ Norm-Trace LRC code. Then we have

$$d_2(C) \geq q^{2u-1} + q^{u-1} - tq^{u-1} - (q^{u-1}-1)(1+q+\dots+q^{u-1}).$$

- Let $t \geq 1$ and let C be a Norm-Trace LRC code with parameters $(q^{2u-1}-q^{u-1}, (t+1)(q+\dots+q^{u-1}), q+\dots+q^{u-1})$. Then we have

$$d_2(C) \geq q^{2u-1} - (t-1)q^{u-1} - (1+q^{u-1})(q+\dots+q^{u-1}).$$

5 HERMITIAN LRC CODES

In this section we improve the bound on the distance of Hermitian LRC codes proposed in [1] using some properties of *Hermitian codes* which are a special case of algebraic geometric codes.

5.1 Hermitian codes

Let us consider $K = \mathbb{F}_{q^2}$ a finite field with q^2 elements. The *Hermitian curve* \mathcal{H} is defined over K by the affine equation

$$x^{q+1} = y^q + y \text{ where } x, y \in K. \quad (2)$$

This curve has genus $g = \frac{q(q-1)}{2}$ and has $q^3 + 1$ points of degree one, namely a pole Q_∞ and $n = q^3$ rational affine points, denoted by $\mathcal{P}_{\mathcal{H}} = \{P_1, \dots, P_n\}$ [31].

Definition 4. Let $m \in \mathbb{N}$ such that $0 \leq m \leq q^3 + q^2 - q - 2$. Then the **Hermitian code** $C(m, q)$ is the code $C(D, mQ_\infty)$ where

$$D = \sum_{\alpha^{q+1} = \beta^q + \beta} P_{\alpha, \beta}$$

is the sum of all places of degree one (except Q_∞ , that is a point at infinity) of the Hermitian function field $K(\mathcal{H})$.

By Lemma 6.4.4. of [33] we have that

$$\mathcal{B}_{m, q} = \{x^i y^j \mid qi + (q+1)j \leq m, 0 \leq i \leq q^2 - 1, 0 \leq j \leq q - 1\},$$

forms a basis of $\mathcal{L}(mQ_\infty)$. For this reason, the Hermitian code $C(m, q)$ could be seen as $\text{Span}_{\mathbb{F}_{q^2}} \langle \text{ev}_{\mathcal{P}_{\mathcal{H}}}(\mathcal{B}_{m, q}) \rangle$. Moreover, the dual of $C(m, q)$ denoted by $C(m_\perp, q) = C^\perp(m, q)$ is again an Hermitian code and it is well known (Proposition 8.3.2 of [33]) that the degree m of the divisor has the following relation with respect to m_\perp :

$$m_\perp = n + 2g - 2 - m. \quad (3)$$

The Hermitian codes can be divided in four phases [11], any of them having specific explicit formulas linking their dimension and their distance [22]. In particular we are interested in the first and the last phase of Hermitian codes, which are:

I PHASE: $0 \leq m_{\perp} \leq q^2 - 2$. Then we have $m_{\perp} = aq + b$ where $0 \leq b \leq a \leq q - 1$ and $b \neq q - 1$. In this case, the distance is

$$\begin{cases} d = a + 1 & \text{if } a > b \\ d = a + 2 & \text{if } a = b. \end{cases} \quad (4)$$

IV PHASE: $n - 1 \leq m_{\perp} \leq n + 2g - 2$. In this case $m_{\perp} = n + 2g - 2 - aq - b$ where a, b are integers such that $0 \leq b \leq a \leq q - 2$ and the distance is

$$d = n - aq - b. \quad (5)$$

5.2 Bound on distance of Hermitian LRC codes

Let $K = \mathbb{F}_{q^2}$ be a finite field, where q is a power of a prime. Let $\mathcal{X} = \mathcal{H}$ be the Hermitian curve with affine equation as in (2). We recall that this curve has $q^3 \mathbb{F}_{q^2}$ -rational affine points plus one at infinity, that we denoted by Q_{∞} .

We consider two of the three constructions of Hermitian LRC codes proposed in [1] and we improve the bound on distance of Hermitian LRC codes using properties of Hermitian codes. In particular, if we find an Hermitian code $C(m, q) = C_{\text{Her}}$ such that $C_{\text{LRC}} \subset C_{\text{Her}}$, then we have $d_{\text{LRC}} \geq d_{\text{Her}}$.

PROJECTION ON \mathcal{X} By Proposition 4 of [1], we have a family of (n, k, r) Hermitian LRC codes with $r = q - 1$, length $n = q^3$, dimension $k = (t - 1)(q - 1)$ and distance $d \geq n - tq - (q - 2)(q + 1)$. Moreover, for these codes, $S = K$, $D = tQ_{\infty}$ for some $1 \leq t \leq q^2 - 1$ and the basis for the vector space V is

$$\mathcal{B} = \{x^j y^i \mid j = 0, \dots, t, i = 0, \dots, q - 2\}. \quad (6)$$

Using the Hermitian codes, we improve the bound on the distance for any integer t , such that $q^2 - q + 1 \leq t \leq q^2 - 1$.

To find an Hermitian code $C(m, q) = C_{\text{Her}}$ such that $C_{\text{LRC}} \subset C_{\text{Her}}$, we have to compute the set $\mathcal{B}_{m, q}$, that is, we have to find m . After that, to compute the distance of $C(m, q)$ we use (4) and (5).

We consider the first Hermitian phase: $0 \leq m_{\perp} \leq q^2 - 2$, that is, $q^2 - q + 1 \leq t \leq q^2 - 1$.

For this phase $m_{\perp} = aq + b$, where $0 \leq b \leq a \leq q - 1$ and the distance of the Hermitian code is either $d = a + 1$ if $a > b$ or $d = a + 2$ if $a = b$. By (6), m must be equal to $m = qt + (q + 1)(q - 2)$ and by (3) we have that $m_{\perp} = n + 2g - 2 - m = q(q^2 - t)$. So $b = 0$ and $a = q^2 - t$ and the distance of the Hermitian code is $d_{\text{Her}} = a + 1 = q^2 - t + 1$, since $a > b$. This implies that

$$d_{\text{LRC}} \geq q^2 - t + 1, \text{ for any } t \geq q^2 - q + 1. \quad (7)$$

Note that (7) improves the bound on the distance proposed in Proposition 4 of [1] since

$$q^2 - t + 1 > q^3 - tq - (q - 2)(q + 1) \iff t(q - 1) > q(q - 1)^2 + 1 \iff t > q^2 - q.$$

We just proved the following:

Proposition 3. *Let $q^2 - q + 1 \leq t \leq q^2 - 1$. It is possible to construct a family of (n, k, r) Hermitian LRC codes $\{C_t\}_{q^2 - q + 1 \leq t \leq q^2 - 1}$ with the following parameters:*

$$n = q^3, k = (t - 1)(q - 1), r = q - 1 \text{ and } d \geq q^2 - t + 1.$$

TWO RECOVERING SETS In [1] the authors propose an Hermitian code with two recovering sets of size $r_1 = q - 1$ and $r_2 = q$, denoted by LRC(2). They consider

$$L = \text{Span}\{x^i y^j, i = 0, \dots, q - 2, j = 0, \dots, q - 1\}$$

and a linear code C obtained by evaluating the functions in L at the points of $B = g^{-1}(\mathbb{F}_{q^2} \setminus M)$, where $g(x, y) = x$ and $M = \{a \in \mathbb{F}_q \mid a^q + a = 0\}$. So $|B| = q^3 - q$. By Proposition 4.3 of [1], the LRC(2) code has length $n = (q^2 - 1)q$, dimension $k = (q - 1)q$ and distance

$$d \geq (q + 1)(q^2 - 3q + 3) = q^3 - 2q^2 + 3. \quad (8)$$

As before, we improve the bound on the distance using Hermitian codes that contains the LRC(2) code. To do this we have to find m_\perp . By L , we have that $m = q(q - 1) + (q + 1)(q - 2)$ so we are in the fourth phase of Hermitian codes because $m_\perp = n + 2g - 2 - m = q^3 - q^2 + q$. In this case $d_{\text{Her}} = m_\perp - 2g + 2 = q^3 + 2q + 2$. Since $|B| = q^3 - q$, we have that

$$d_{\text{LRC}} \geq d_{\text{Her}} - q = q^3 + q + 2. \quad (9)$$

Note that this bound improves bound (8). We just proved the following proposition:

Proposition 4. *Let C be a linear code obtained by evaluating the functions in L at the points of B . Then C has the following parameters:*

$$n = (q^2 - 1)q, k = (q - 1)q, r_1 = q - 1, r_2 = q \text{ and } d \geq q^3 + q + 2.$$

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